

# Pointwise analog of the Stečkin approximation theorem

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## Abstract

We show the pointwise version of the Stečkin theorem on approximation by de la Vallée-Poussin means. The result on norm approximation is also derived.

**Key words:** Pointwise approximation by de la Vallée-Poussin means

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# 1 Introduction

Let  $L^p$  ( $1 \leq p < \infty$ )  $[C]$  be the class of all  $2\pi$ -periodic real-valued functions integrable in the Lebesgue sense with  $p$ -th power [continuous] over  $Q = [-\pi, \pi]$  and let  $X^p = L^p$  when  $1 \leq p < \infty$  or  $X^p = C$  when  $p = \infty$ .

Let us define the norms of  $f \in X^p$  as

$$\|f\| = \|f\|_{X^p} = \|f(\cdot)\|_{X^p} := \begin{cases} \left\{ \int_Q |f(x)|^p dx \right\}^{\frac{1}{p}} & \text{when } 1 \leq p < \infty \\ \sup_{x \in Q} |f(x)| & \text{when } p = \infty \end{cases}$$

and

$$\begin{aligned} \|f\|_{x,\delta} &= \|f\|_{X^p,x,\delta} = \|f(\cdot)\|_{X^p,x,\delta} := \sup_{0 < h \leq \delta} \|f(\cdot)\|_{X^p,x,h}^\circ \\ &= \begin{cases} \sup_{0 < h \leq \delta} \left\{ \frac{1}{2h} \int_{x-h}^{x+h} |f(t)|^p dt \right\}^{\frac{1}{p}} & \text{when } 1 \leq p < \infty \\ \sup_{0 < h \leq \delta} \left\{ \sup_{0 < |t| \leq h} |f(x+t)| \right\} & \text{when } p = \infty \end{cases} \end{aligned}$$

where

$$\begin{aligned} \|f\|_{x,\delta}^\circ &= \|f\|_{X^p,x,\delta}^\circ = \|f(\cdot)\|_{X^p,x,\delta}^\circ \\ &= \begin{cases} \left\{ \frac{1}{2h} \int_{x-\delta}^{x+\delta} |f(t)|^p dt \right\}^{\frac{1}{p}} & \text{when } 1 \leq p < \infty \\ \sup_{0 < |t| \leq h} |f(x+h)| & \text{when } p = \infty \end{cases} \quad (\delta > 0). \end{aligned}$$

We note additionally that

$$\|f\|_{X,x,0} = \|f\|_{X,x,0}^\circ = |f(x)|.$$

Consider the trigonometric Fourier series of  $f$

$$Sf(x) = \frac{a_0(f)}{2} + \sum_{k=0}^{\infty} (a_k(f) \cos kx + b_k(f) \sin kx)$$

with the partial sums  $S_k f$ .

Let

$$\sigma_{n,m} f(x) := \frac{1}{m+1} \sum_{k=n-m}^n S_k f(x) \quad (m \leq n = 0, 1, 2, \dots)$$

As a measure of approximation by the above quantities we use the pointwise

characteristics

$$\begin{aligned}
w_x f(\delta) &= w_x f(\delta)_{X^p} := \|\Delta_x f(\cdot)\|_{X^p, x, \delta} \\
&= \begin{cases} \sup_{0 < h \leq \delta} \left\{ \frac{1}{2h} \int_{-h}^h |\Delta_x f(t)|^p dt \right\}^{1/p} & \text{when } 1 \leq p < \infty \\ \sup_{0 < h \leq \delta} \left\{ \sup_{0 < |t| \leq h} |\Delta_x f(t)| \right\} & \text{when } p = \infty \end{cases}
\end{aligned}$$

cf. [1] and

$$\begin{aligned}
w_x^\circ f(\delta) &= w_x^\circ f(\delta)_{X^p} := \|\Delta_x f(\cdot)\|_{X^p, x, \delta}^\circ \\
&= \begin{cases} \left\{ \frac{1}{2\delta} \int_{-\delta}^{\delta} |\Delta_x f(t)|^p dt \right\}^{1/p} & \text{when } 1 \leq p < \infty \\ \sup_{0 < |t| \leq \delta} |\Delta_x f(t)| & \text{when } p = \infty \end{cases}
\end{aligned}$$

and also

$$\Omega_x f\left(\frac{\pi}{n+1}\right) = \Omega_x f\left(\frac{\pi}{n+1}\right)_{X^p} := \frac{1}{n+1} \sum_{k=0}^n w_x f\left(\frac{\pi}{k+1}\right)_{X^p}$$

and

$$\Omega_x^\circ f\left(\frac{\pi}{n+1}\right) = \Omega_x^\circ f\left(\frac{\pi}{n+1}\right)_{X^p} = \frac{1}{n+1} \sum_{k=0}^n w_x^\circ f\left(\frac{\pi}{k+1}\right)_{X^p},$$

$$\text{where } \Delta_x f(t) := f(x+t) - f(x),$$

constructed on the base of definition of  $X^p$ -points ([Lebesgue points( $L^p$  - points)] or [points of continuity ( $C$  - points)]). We also use the modulus of continuity of  $f$  in the space  $X^p$  defined by the formula

$$\omega f(\delta) = \omega f(\delta)_{X^p} := \sup_{0 < |h| \leq \delta} \|\Delta_x f(h)\|_{X^p}$$

and its arithmetic mean

$$\Omega f\left(\frac{\pi}{n+1}\right) = \Omega f\left(\frac{\pi}{n+1}\right)_{X^p} = \frac{1}{n+1} \sum_{k=0}^n \omega f\left(\frac{\pi}{k+1}\right)_{X^p}.$$

We can observe that, for  $f \in X^{\tilde{p}}$  and  $\tilde{p} \geq p$ ,

$$\|w_{\cdot} f(\delta)_{X^p}\|_C \leq \omega f(\delta)_C,$$

whence

$$\|\Omega_{\cdot} f(\delta)_{X^p}\|_C \leq \Omega f(\delta)_C$$

and

$$\|w_{\cdot}^\circ f(\delta)_{X^p}\|_{X^p} \leq \omega f(\delta)_{X^p},$$

whence

$$\|\Omega_{\cdot}^{\circ} f(\delta)_{X^p}\|_{X^p} \leq \Omega f(\delta)_{X^p}.$$

Let introduce one more measure of pointwise approximation analogical to the best approximation of function  $f$  by trigonometric polynomials  $T$  of the degree at most  $n$  ( $T \in H_n$ )

$$E_n(f)_{X^p} := \inf_{T \in H_n} \{\|f(\cdot) - T(\cdot)\|_{X^p}\},$$

namely

$$\begin{aligned} E_n(f, x; \delta) &= E_n(f, x; \delta)_{X^p} := \inf_{T \in H_n} \left\{ \|f(\cdot) - T(\cdot)\|_{X^p, x, \delta} \right\} \\ &= \begin{cases} \inf_{T \in H_n} \left\{ \sup_{0 < h \leq \delta} \left[ \frac{1}{2h} \int_{-h}^h |f(x+t) - T(x+t)|^p dt \right]^{\frac{1}{p}} \right\} & \text{when } 1 \leq p < \infty \\ \inf_{T \in H_n} \left\{ \sup_{0 < |h| \leq \delta} |f(x+h) - T(x+h)| \right\} & \text{when } p = \infty \end{cases} \end{aligned}$$

and

$$E_n^{\circ}(f, x; \delta) = E_n^{\circ}(f, x; \delta)_{X^p} := \inf_{T \in H_n} \left\{ \|f(\cdot) - T(\cdot)\|_{X^p, x, \delta}^{\circ} \right\}.$$

We will also use its arithmetic mean

$$F_{n,m}(f, x) = F_{n,m}(f, x)_X := \frac{1}{m+1} \sum_{k=0}^m E_n \left( f, x; \frac{\pi}{k+1} \right)_X$$

and

$$F_{n,m}^{\circ}(f, x) = F_{n,m}^{\circ}(f, x)_{X^p} := \frac{1}{m+1} \sum_{k=0}^m E_n^{\circ} \left( f, x; \frac{\pi}{k+1} \right)_{X^p}.$$

Denote also

$$X^p(w_x) = \{f \in X^p : w_x f(\delta) \leq w_x(\delta)\},$$

where  $w_x$  is a function of modulus of continuity type on the interval  $[0, +\infty)$ , i.e. a nondecreasing continuous function having the following properties:  $w_x(0) = 0$ ,  $w_x(\delta_1 + \delta_2) \leq w_x(\delta_1) + w_x(\delta_2)$  for any  $0 \leq \delta_1 \leq \delta_2 \leq \delta_1 + \delta_2$ .

Using these characteristics we will show the pointwise version of the Stečkin [5] generalization of the Fejér-Lebesgue theorem. As a corollaries we will obtain the mentioned original result of S. B. Stečkin on norm approximation as well the result of N. Tanović-Miller [6].

By  $K$  we shall designate either an absolute constant or a constant depending on some parameters, not necessarily the same of each occurrence.

## 2 Statement of the results

At the begin we formulate the partial solution of the considered problem.

**Theorem 1** [4] *If  $f \in X^p$  then, for any positive integer  $m \leq n$  and all real  $x$ ,*

$$\begin{aligned} |\sigma_{n,m}f(x) - f(x)| &\leq \pi^2 E_{n-m}^\circ(f, x, \frac{\pi}{2n-m+1})_X + 6F_{n-m,m}^\circ(f, x)_X \\ &\quad + \int_{\frac{\pi}{m+1}}^{\frac{\pi}{2n-m+1}} \frac{E_{n-m}^\circ(f, x, t)_X}{t} dt + E_{n-m}^\circ(f, x; 0)_{X^p} \end{aligned}$$

and

$$\begin{aligned} |\sigma_{n,m}f(x) - f(x)| &\leq (6 + \pi^2) F_{n-m,m}(f, x)_{X^p} \left[ 1 + \ln \frac{n+1}{m+1} \right] \\ &\quad + E_{n-m}(f, x; 0)_{X^p}. \end{aligned}$$

Now, we can present the main result on pointwise approximation.

**Theorem 2** *If  $f \in X^p$  then, for any positive integer  $m \leq n$  and all real  $x$ ,*

$$|\sigma_{n,m}f(x) - f(x)| \leq K \sum_{\nu=0}^n \frac{F_{n-m+\nu,m}(f, x)_{X^p} + F_{n-m+\nu,\nu}(f, x)_{X^p}}{m + \nu + 1} + E_{2n}(f, x; 0)_{X^p}.$$

This immediately yields the following result of Stečkin [5]

**Theorem 3** *If  $f \in C$  then, for any positive integer  $n$  and  $m \leq n$*

$$\|\sigma_{n,m}f(\cdot) - f(\cdot)\|_C \leq K \sum_{\nu=0}^n \frac{E_{n-m+\nu}(f)_C}{m + \nu + 1}.$$

**Remark 1** *Theorem also holds if instead of  $C$  we consider the spaces  $X^p$  with  $1 < p < \infty$ . In the proof we need the Hardy-Littlewood estimate of the maximal function.*

At every  $X^p$  - point  $x$  of  $f$

$$\Omega_x f(\gamma)_{X^p} = o_x(1) \quad \text{as } \gamma \rightarrow 0+$$

and thus from Theorem 1 we obtain the corollary which state the result of the Tanović-Miller type[6].

**Corollary 1** *If  $f \in X^p$  then, for any positive integer  $m \leq n$  at every  $X^p$  - point  $x$  of  $f$ ,*

$$|\sigma_{n,m}f(x) - f(x)| = o_x(1) \left[ 1 + \ln \frac{n+1}{m+1} \right] \quad \text{as } n \rightarrow \infty.$$

### 3 Auxiliary results

In order to proof our theorems we require some lemmas

**Lemma 1** *If  $T_n$  is the trigonometric polynomial of the degree at most  $n$  of the best approximation of  $f \in X^p$  with respect to the norm  $\|\cdot\|_{X^p}$  then, it is also the trigonometric polynomial of the degree at most  $n$  of the best approximation of  $f \in X^p$  with respect to the norm  $\|\cdot\|_{X^p, x, \delta}$  for any  $\delta \in [0, \pi]$ .*

**Proof.** From the inequalities

$$\begin{aligned} \|E_n(f, \cdot, \delta)_{X^p}\|_{X^p} &\geq \|E_n^\circ(f, \cdot, \delta)_{X^p}\|_{X^p} \\ &= \left\| \|f - T_{n, \delta}\|_{X^p, \cdot, \delta}^\circ \right\|_{X^p} = \|f - T_{n, \delta}\|_{X^p} \\ &\geq \|f - T_n\|_{X^p} = E_n(f)_{X^p} \end{aligned}$$

and

$$\|E_n^\circ(f, \cdot, \delta)_{X^p}\|_{X^p} \leq \left\| \|f - T_n\|_{X^p, \cdot, \delta}^\circ \right\|_{X^p} = \|f - T_n\|_{X^p} = E_n(f)_{X^p} ,$$

where  $T_{n, \delta}$  and  $T_n$  are the trigonometric polynomials of the degree at most  $n$  of the best approximation of  $f \in X^p$  with respect to the norms  $\|\cdot\|_{X^p, x, \delta}^\circ$  and  $\|\cdot\|_{X^p}$  respectively, we obtain relation

$$\|f - T_{n, \delta}\|_{X^p} = \|f - T_n\|_{X^p} = E_n(f)_{X^p} ,$$

whence  $T_{n, \delta} = T_n$  for any  $\delta \in [0, \pi]$  by uniqueness of the trigonometric polynomial of the degree at most  $n$  of the best approximation of  $f \in X^p$  with respect to the norm  $\|\cdot\|_{X^p}$  (see e.g. [2] p. 96). We can also observe that for such  $T_n$  and any  $h \in [0, \delta]$

$$\|f - T_n\|_{X^p, x, h}^\circ = E_n^\circ(f, x, h)_{X^p} \leq E_n(f, x, \delta)_{X^p} \leq \|f - T_n\|_{X^p, x, \delta} .$$

Hence

$$E_n(f, x, \delta)_{X^p} = \|f - T_n\|_{X^p, x, \delta}$$

and our proof is complete. ■

**Lemma 2** *If  $n \in \mathbb{N}_0$  and  $\delta > 0$  then  $E_n(f, x; \delta)_{X^p}$  is nonincreasing function of  $n$  and nondecreasing function of  $\delta$ . These imply that for  $m, n \in \mathbb{N}$  the function  $F_{n, m}(f, x)_{X^p}$  is nonincreasing function of  $n$  and  $m$  simultaneously.*

**Proof.** The first part of our statement follows from the property of the norm  $\|\cdot\|_{x,\delta}$  and supremum. The second part is a consequence of the calculation

$$\begin{aligned}\frac{F_{n,m+1}(f,x)_{X^p}}{F_{n,m}(f,x)_{X^p}} &= \frac{m+1}{m+2} \left( 1 + \frac{E_n\left(f,x;\frac{\pi}{m+2}\right)_{X^p}}{\sum_{k=0}^m E_n\left(f,x;\frac{\pi}{k+1}\right)_{X^p}} \right) \\ &\leq \frac{m+1}{m+2} \left( 1 + \frac{E_n\left(f,x;\frac{\pi}{m+1}\right)_{X^p}}{\sum_{k=0}^m E_n\left(f,x;\frac{\pi}{k+1}\right)_{X^p}} \right) \\ &= \frac{m+1}{m+2} \left( 1 + \frac{1}{m+1} \right) = 1.\end{aligned}$$

■

**Lemma 3** Let  $m, n, q \in \mathbb{N}_0$  such that  $m \leq n$  and  $q \geq m+1$ . If  $f \in X^p$  then

$$|\sigma_{n+q,m}f(x) - \sigma_{n,m}f(x)| \leq K F_{n-m,m}(f,x)_{X^p} \sum_{\nu=0}^{q-1} \frac{1}{m+\nu+1}.$$

**Proof.** It is clear that

$$\begin{aligned}\sigma_{n,m}f(x) &= \frac{1}{m+1} \sum_{k=n-m}^n \frac{1}{\pi} \int_{-\pi}^{\pi} f(x+t) D_k(t) dt \\ &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x+t) V_{n,m}(t) dt\end{aligned}$$

where

$$V_{n,m}(t) = \frac{1}{m+1} \sum_{k=n-m}^n D_k(t) \quad \text{and} \quad D_k(t) = \frac{\sin \frac{(2k+1)t}{2}}{2 \sin \frac{t}{2}}.$$

Hence, by orthogonality of the trigonometric system,

$$\begin{aligned}&\sigma_{n+q,m}f(x) - \sigma_{n,m}f(x) \\ &= \frac{1}{\pi} \int_{-\pi}^{\pi} [f(x+t) - T_{n-m}(x+t)] (V_{n+q,m}(t) - V_{n,m}(t)) dt \\ &= \frac{1}{\pi(m+1)} \sum_{k=n-m}^n \int_{-\pi}^{\pi} [f(x+t) - T_{n-m}(x+t)] (D_{k+q}(t) - D_k(t)) dt \\ &= \frac{1}{\pi(m+1)} \sum_{k=n-m}^n \int_{-\pi}^{\pi} [f(x+t) - T_{n-m}(x+t)] \\ &\quad \frac{\sin \frac{(2k+2q+1)t}{2} - \sin \frac{(2k+1)t}{2}}{2 \sin \frac{t}{2}} dt \\ &= \frac{1}{\pi(m+1)} \sum_{k=n-m}^n \int_{-\pi}^{\pi} [f(x+t) - T_{n-m}(x+t)] \frac{\sin \frac{qt}{2} \cos \frac{(2k+q+1)t}{2}}{\sin \frac{t}{2}} dt\end{aligned}$$

with trigonometric polynomial  $T_{n-m}$  of the degree at most  $n-m$  of the best approximation of  $f$ .

Using the notations

$$\begin{aligned} I_1 &= \left[-\frac{\pi}{q}, \frac{\pi}{q}\right], & I_2 &= \left[-\frac{\pi}{m+1}, -\frac{\pi}{q}\right] \cup \left[\frac{\pi}{q}, \frac{\pi}{m+1}\right] \\ I_3 &= \left[-\pi, -\frac{\pi}{m+1}\right] \cup \left[\frac{\pi}{m+1}, \pi\right] \end{aligned}$$

we get

$$\begin{aligned} \sum &= \frac{1}{\pi(m+1)} \sum_{k=n-m}^n \left( \int_{I_1} + \int_{I_2} + \int_{I_3} \right) [f(x+t) - T_{n-m}(x+t)] \\ &\quad \frac{\sin \frac{qt}{2} \cos \frac{(2k+q+1)t}{2}}{\sin \frac{t}{2}} dt \\ &= \sum_1 + \sum_2 + \sum_3. \end{aligned}$$

and

$$\begin{aligned} \sum_1 &\leq \frac{1}{\pi(m+1)} \sum_{k=n-m}^n \int_{I_1} |f(x+t) - T_{n-m}(x+t)| q dt \\ &= \frac{q}{\pi} \int_{I_1} |f(x+t) - T_{n-m}(x+t)| dt \\ &\leq 2E_{n-m} \left( f, x; \frac{\pi}{q} \right)_{X^p} \end{aligned}$$

We next evaluate the sums  $\sum_2$  and  $\sum_3$  using the partial integrating and Lemma 1. Thus

$$\begin{aligned} \sum_2 &\leq \int_{I_2} \frac{|f(x+t) - T_{n-m}(x+t)|}{t} dt \\ &= 2 \left[ \frac{1}{2t} \int_{-t}^t |f(x+u) - T_{n-m}(x+u)| du \right]_{t=\frac{\pi}{q}}^{t=\frac{\pi}{m+1}} \\ &\quad + 2 \int_{\frac{\pi}{q}}^{\frac{\pi}{m+1}} \frac{1}{t} \left[ \frac{1}{2t} \int_{-t}^t |f(x+u) - T_{n-m}(x+u)| du \right] dt \\ &\leq 2E_{n-m} \left( f, x; \frac{\pi}{m+1} \right)_{X^p} + 2 \int_{-\frac{\pi}{q}}^{\frac{\pi}{q}} \frac{1}{t} E_{n-m}(f, x; t)_{X^p} dt \\ &\leq 4E_{n-m} \left( f, x; \frac{\pi}{m+1} \right)_{X^p} \left[ 1 + \ln \frac{q}{m+1} \right] \\ &\leq 4E_{n-m} \left( f, x; \frac{\pi}{m+1} \right)_{X^p} \left[ 1 + \sum_{\nu=0}^{q-1} \frac{1}{m+\nu+1} \right] \end{aligned}$$



and

$$\begin{aligned}
\Sigma_3 &\leq \frac{1}{m+1} \int_{I_3} \frac{|f(x+t) - T_{n-m}(x+t)|}{t} \\
&\quad \left| \sum_{k=n-m}^n \cos\left(kt + \frac{q+1}{2}t\right) \right| dt \\
&\leq \frac{1}{m+1} \int_{I_3} \frac{|f(x+t) - T_{n-m}(x+t)|}{t} \\
&\quad \left| \frac{2 \sin \frac{(n+1)t}{2} \cos \frac{(2n-m+q+1)t}{2}}{2 \sin \frac{t}{2}} \right| dt \\
&\leq \frac{\pi}{m+1} \int_{I_3} \frac{|f(x+t) - T_{n-m}(x+t)|}{t^2} dt \\
&= \frac{\pi}{m+1} \left\{ 2 \left[ \frac{1}{2t} \int_{-t}^t |f(x+u) - T_{n-m}(x+u)| du \right]_{t=\frac{\pi}{m+1}}^{t=\pi} \right. \\
&\quad \left. + 4 \int_{\frac{\pi}{m+1}}^{\pi} \frac{1}{t^2} \left[ \frac{1}{2t} \int_{-t}^t |f(x+u) - T_{n-m}(x+u)| du \right] dt \right\} \\
&\leq \frac{\pi}{m+1} \left\{ 2E_{n-m}(f, x; \pi)_{X^p} + 4 \int_{\frac{\pi}{m+1}}^{\pi} \frac{1}{t^2} E_{n-m}(f, x; t)_{X^p} dt \right\} \\
&= \frac{2\pi}{m+1} \left\{ E_{n-m}(f, x; \pi)_{X^p} + 2 \int_1^{m+1} \frac{E_{n-m}\left(f, x; \frac{\pi}{u}\right)_{X^p} \pi du}{\pi^2/u^2} \frac{\pi}{u^2} \right\} \\
&= \frac{2\pi}{m+1} \left\{ E_{n-m}(f, x; \pi)_{X^p} + \frac{2}{\pi} \sum_{k=0}^{m-1} \int_{k+1}^{k+2} E_{n-m}\left(f, x; \frac{\pi}{u}\right)_{X^p} du \right\} \\
&= \frac{2\pi}{m+1} \left\{ E_{n-m}(f, x; \pi)_{X^p} + \frac{2}{\pi} \sum_{k=0}^{m-1} E_{n-m}\left(f, x; \frac{\pi}{k+1}\right)_{X^p} \right\} \\
&\leq \frac{2\pi + \frac{2}{\pi}}{m+1} \sum_{k=0}^{m-1} E_{n-m}\left(f, x; \frac{\pi}{k+1}\right)_{X^p}
\end{aligned}$$

which proves Lemma 2. ■

Before formulating the next lemmas we define a new difference. Let  $m, n \in \mathbb{N}_0$  and  $m \leq n$ . Denote

$$\tau_{n,m}f(x) := (m+1) \{ \sigma_{n+m+1,m}f(x) - \sigma_{n,m}f(x) \}.$$

**Lemma 4** Let  $m, n, \mu \in \mathbb{N}_0$  such that  $2\mu \leq m \leq n$ . If  $f \in X^p$  then

$$|\tau_{n,m}f(x) - \tau_{n-\mu,m-\mu}f(x)| \leq K\mu F_{n-\mu+1,\mu-1}(f, x)_{X^p} \ln \frac{m}{\mu}.$$

**Proof.** The proof follows by the method of Leindler[3]. Namely

$$\tau_{n,m}f(x) - \tau_{n-\mu,m-\mu}f(x) = \left( \sum_{k=n+m-2\mu+2}^{n+m+1} - 2 \sum_{k=n-\mu+1}^n \right) [S_k f(x) - f(x)]$$

and

$$\begin{aligned} & |\tau_{n,m}f(x) - \tau_{n-\mu,m-\mu}f(x)| \leq \\ & \leq \left| \left( \sum_{k=n+m-2\mu+2}^{n+m+1} - \sum_{k=n-\mu+1}^n \right) [S_k f(x) - f(x)] \right| \\ & + \left| \left( \sum_{k=n+m-\mu+2}^{n+m+1} - \sum_{k=n-\mu+1}^n \right) [S_k f(x) - f(x)] \right| \\ & = \mu |\sigma_{n+m-\mu+1,\mu-1}f(x) - \sigma_{n,\mu-1}f(x)| \\ & + \mu |\sigma_{n+m+1,\mu-1}f(x) - \sigma_{n,\mu-1}f(x)|. \end{aligned}$$

By Lemma 2, for  $2\mu \leq m$ ,

$$\begin{aligned} & |\tau_{n,m}f(x) - \tau_{n-\mu,m-\mu}f(x)| \leq \\ & \leq K\mu F_{n-\mu+1,\mu-1}(f, x)_{X^p} \left[ 1 + \ln \frac{(n-\mu+1) + \mu - 1}{\mu} \right] \\ & + K\mu F_{n-\mu+1,\mu-1}(f, x)_{X^p} \left[ 1 + \ln \frac{m + \mu - 1}{\mu} \right] \\ & \leq K\mu F_{n-\mu+1,\mu-1}(f, x)_{X^p} \left[ 1 + \ln \frac{m}{\mu} \right] \end{aligned}$$

and our proof is complete. ■

**Lemma 5** Let  $m, n \in \mathbb{N}_0$  and  $m \leq n$ . If  $f \in X^p$  then

$$|\tau_{n,m}f(x)| \leq K \sum_{k=n-m}^n F_{k,k-n+m}(f, x)_{X^p}.$$

**Proof.** Our proof runs parallel with the proof of Theorem 1 in [5].

If  $m = 0$  then

$$|\tau_{n,0}f(x)| = |\sigma_{n+1,0}f(x) - \sigma_{n,0}f(x)| \leq K F_{n,0}(f, x)_{X^p}.$$

and if  $m = 1$  then

$$\begin{aligned} |\tau_{n,1}f(x)| &\leq 2|\sigma_{n+1,1}f(x) - \sigma_{n,1}f(x)| \leq KF_{n-1,1}(f, x)_{X^p} \\ &\leq K[F_{n-1,1}(f, x)_{X^p} + F_{n-1,1}(f, x)_{X^p}] \\ &\leq K[F_{n-1,1}(f, x)_{X^p} + F_{n,1}(f, x)_{X^p}] \end{aligned}$$

by Lemma 2 and Lemma 3

Next we construct the same decreasing sequence  $(m_s)$  of integers that was given by S. B. Stečkin. Let

$$m_0 = m, \quad m_s = m_{s-1} - \left\lfloor \frac{m_{s-1}}{2} \right\rfloor \quad (s = 1, 2, \dots)$$

where  $[y]$  denotes the integral part of  $y$ . It is clear that there exists a smallest index  $t \geq 1$  such that  $m_t = 1$  and

$$m = m_0 > m_1 > \dots > m_t = 1.$$

By the definition of the numbers  $m_s$  we have

$$\begin{aligned} m_s &\geq m_{s-1}/2 \\ m_{s-1} - m_s &= \left\lfloor \frac{m_{s-1}}{2} \right\rfloor \geq \left\lfloor \frac{m_{s-1}}{3} \right\rfloor \quad (s = 1, 2, \dots, t) \end{aligned}$$

whence

$$m_{t-1} = 2, \quad m_{t-1} - m_t = 1$$

and

$$m_{s-1} - m_s \leq m_s \leq 3(m_s - m_{s+1}) \quad (s = 1, 2, \dots, t-1)$$

follow.

Under these notations we get the following equality

$$\tau_{n,m}f(x) = \sum_{s=1}^t (\tau_{n-m+m_{s-1}, m_{s-1}}f(x) - \tau_{n-m+m_s, m_s}f(x)) + \tau_{n-m+m_t, m_t}f(x)$$

whence, by  $m_t = 1$ ,

$$|\tau_{n,m}f(x)| \leq \sum_{s=1}^t |\tau_{n-m+m_{s-1}, m_{s-1}}f(x) - \tau_{n-m+m_s, m_s}f(x)| + |\tau_{n-m+m_t, m_t}f(x)|$$

follows.

It is easy to see that the terms in the sum  $\sum_{s=1}^t$ , by Lemma 4, with  $\mu = m_{s-1} - m_s$  and  $m = m_{s-1}$  do not exceed

$$K(m_{s-1} - m_s)F_{n-m+m_s+1, m_{s-1}-m_s}(f, x)_{X^p} \ln \frac{m_{s-1}}{m_{s-1} - m_s},$$

where  $(s = 1, 2, \dots, t-1)$ .

and by Lemma 3 we get

$$|\tau_{n-m+1,1} f(x)| \leq 2 |\sigma_{n-m+2,1} f(x) - \sigma_{n-m+1,1} f(x)| \leq K F_{n-m,1}(f, x)_{X^p}$$

Thus

$$\begin{aligned} |\tau_{n,m} f(x)| &\leq K \sum_{s=1}^{t-1} 3(m_s - m_{s+1}) F_{n-m+m_s+1, m_s}(f, x)_{X^p} \ln 3 \\ &\quad + K F_{n-m+2, m-2}(f, x)_{X^p} + K F_{n-m,1}(f, x)_{X^p} \end{aligned}$$

whence, by the monotonicity of  $F_{\nu, \mu}(f, x)_{X^p}$ ,

$$\begin{aligned} &|\tau_{n,m} f(x)| \\ &\leq K \left( \sum_{s=1}^{t-1} \sum_{\nu=m_{s+1}+1}^{m_s} F_{n-m+\nu+1, \nu}(f, x)_{X^p} + \sum_{\nu=0}^2 F_{n-m+\nu, m-\nu-1}(f, x)_{X^p} \right) \\ &\quad + K F_{n-m,1}(f, x)_{X^p} \\ &\leq K \sum_{\nu=0}^{m_1+1} F_{n-m+\nu, \nu}(f, x)_{X^p} + K F_{n-m,1}(f, x)_{X^p} \\ &\leq K \sum_{\nu=0}^m F_{n-m+\nu, \nu}(f, x)_{X^p} + K F_{n-m,1}(f, x)_{X^p} \\ &\leq K \sum_{k=n-m}^n F_{k, k-n+m}(f, x)_{X^p} + K F_{n-m,1}(f, x)_{X^p} . \end{aligned}$$

■

## 4 Proofs of the results

### 4.1 Proof of Theorem 2

The proof follows the lines of the proofs of Theorem 4 in [5] and Theorem in [3]. Therefore let  $n > 0$  and  $m \leq n$  be fixed. Let us define an increasing sequence  $(n_s : s = 0, 1, \dots, t)$  of indices introduced by S. B. Stečki in the following way. Set  $n_0 = n$ . Assuming that the numbers  $n_0, \dots, n_s$  are already defined and  $n_s < 2n$ , we define  $n_{s+1}$  as follows: Let  $\nu_s$  denote the smallest natural number such that

$$F_{n_s-m+\nu_s, \nu_s}(f, x)_{X^p} \leq \frac{1}{2} F_{n_s-m, \nu_s}(f, x)_{X^p} \quad (\nu = 0, 1, \dots, n).$$

According to the magnitude of  $\nu_s$  we define

$$n_{s+1} = \begin{cases} n_s - m + 1 & \text{for } \nu_s \leq m, \\ n_s + \nu_s & \text{for } m + 1 \leq \nu_s < 2n + m - n_s, \\ 2n + m & \text{for } \nu_s \geq 2n + m - n_s \end{cases}$$

If  $n_{s+1} < 2n$  we continue the procedure, and if once  $n_{s+1} \geq 2n$  then we stop the construction and define  $t := s + 1$ .

By the above definition of  $(n_s)$  we have the following obvious properties:

$$t \geq 1, \quad n = n_0 < n_1 < \dots < n_t, \quad 2n \leq n_t \leq 2n + m,$$

and

$$n_{s+1} - n_s \geq m + 1 \quad (s = 0, 1, \dots, t-1),$$

and relations

$$F_{n_{s+1}-m, \nu}(f, x)_{X^p} \leq \frac{1}{2} F_{n_s-m, \nu}(f, x)_{X^p} \quad \text{for } s = 0, 1, \dots, t-2,$$

and

$$\frac{1}{2} F_{n_s-m, \nu}(f, x)_{X^p} \leq F_{n_{s+1}-m-1, \nu}(f, x)_{X^p} \quad \text{for } s = 0, 1, \dots, t-1$$

whenever  $n_{s+1} - n_s > m + 1$ .

Let us start with

$$\begin{aligned} |\sigma_{n,m} f(x) - f(x)| &= \sum_{s=0}^{t-1} [|\sigma_{n_s, m} f(x) - f(x)| - |\sigma_{n_{s+1}, m} f(x) - f(x)|] \\ &\quad + |\sigma_{n_t, m} f(x) - f(x)| \\ &\leq \sum_{s=0}^{t-1} |\sigma_{n_{s+1}, m} f(x) - \sigma_{n_s, m} f(x)| + |\sigma_{n_t, m} f(x) - f(x)| \\ &= \sum_{s=0}^{t-1} \left| \frac{1}{m+1} \tau_{n_s, m} f(x) \right| + |\sigma_{n_t, m} f(x) - f(x)|. \end{aligned}$$

Using Theorem 1 and that  $2n \leq n_t \leq 2n + m$  we get

$$\begin{aligned} |\sigma_{n_t, m} f(x) - f(x)| &\leq K F_{n_t-m, m}(f, x)_{X^p} \left[ 1 + \ln \frac{n_t+1}{m+1} \right] + |f(x) - T_{n_t-m}(x)| \\ &\leq K \sum_{\nu=0}^n \frac{F_{n-m+\nu, m}(f, x)_{X^p}}{m+\nu+1} + |f(x) - T_{n_t-m}(x)| \\ &\leq K \sum_{\nu=0}^n \frac{F_{n-m+\nu, m}(f, x)_{X^p} + F_{n-m+\nu, \nu}(f, x)_{X^p}}{m+\nu+1} + |f(x) - T_{n_t-m}(x)| \\ &\leq K \sum_{\nu=0}^n \frac{F_{n-m+\nu, m}(f, x)_{X^p} + F_{n-m+\nu, \nu}(f, x)_{X^p}}{m+\nu+1} + E_{n_t-m}(f, x; 0)_{X^p} \\ &\leq K \sum_{\nu=0}^n \frac{F_{n-m+\nu, m}(f, x)_{X^p} + F_{n-m+\nu, \nu}(f, x)_{X^p}}{m+\nu+1} + E_{2n}(f, x; 0)_{X^p}. \end{aligned}$$

The estimate of the sum we derive from the following one

$$\left| \frac{1}{m+1} \tau_{n_s, m} f(x) \right| \leq K \sum_{\nu=0}^{n_{s+1}-n_s-1} \frac{F_{n_s-m+\nu, m}(f, x)_{X^p} + F_{n_s-m+\nu, \nu}(f, x)_{X^p}}{m+\nu+1}.$$

The proof of this inequality we split in two parts. If  $n_{s+1} - n_s = m + 1$ , then by Lemma 5,

$$\begin{aligned} \left| \frac{1}{m+1} \tau_{n_s, m} f(x) \right| &\leq K \frac{1}{m+1} \sum_{k=n_s-m}^{n_s} F_{k, k-n_s+m}(f, x)_{X^p} \\ &\leq K \sum_{\nu=0}^{n_{s+1}-n_s-1} \frac{F_{n_s-m+\nu, \nu}(f, x)_{X^p}}{m+\nu+1}. \end{aligned}$$

If  $n_{s+1} - n_s > m + 1$ , then, by Lemma 3,

$$\left| \frac{1}{m+1} \tau_{n_s, m} f(x) \right| \leq K F_{n_s-m, m}(f, x)_{X^p} \sum_{\nu=0}^{n_{s+1}-n_s-1} \frac{1}{m+\nu+1}$$

and since  $\frac{1}{2} F_{n_s-m, m}(f, x)_{X^p} \leq F_{n_{s+1}-m-1, m}(f, x)_{X^p}$  we have

$$\begin{aligned} \left| \frac{1}{m+1} \tau_{n_s, m} f(x) \right| &\leq 2K F_{n_{s+1}-m-1, m}(f, x)_{X^p} \sum_{\nu=0}^{n_{s+1}-n_s-1} \frac{1}{m+\nu+1} \\ &\leq 2K \sum_{\nu=0}^{n_{s+1}-n_s-1} \frac{F_{n_s-m+\nu, m}(f, x)_{X^p}}{m+\nu+1} \end{aligned}$$

Consequently,

$$\begin{aligned} &\sum_{s=0}^{t-1} \left| \frac{1}{m+1} \tau_{n_s, m} f(x) \right| \\ &\leq 2K \sum_{s=0}^{t-1} \sum_{\nu=0}^{n_{s+1}-n_s-1} \frac{F_{n_s-m+\nu, m}(f, x)_{X^p} + F_{n_s-m+\nu, \nu}(f, x)_{X^p}}{m+\nu+1} \end{aligned}$$

Since  $n_{s+1} - n_s \leq 2n + m - n - 1 = n + m - 1$  for all  $s \leq t - 1$ , changing the order of summation we get

$$\begin{aligned} &\sum_{s=0}^{t-1} \left| \frac{1}{m+1} \tau_{n_s, m} f(x) \right| \\ &\leq 2K \sum_{\nu=0}^{n+m-1} \frac{1}{m+\nu+1} \sum_{s: n_{s+1}-n_s > \nu} [F_{n_s-m+\nu, m}(f, x)_{X^p} + F_{n_s-m+\nu, \nu}(f, x)_{X^p}]. \end{aligned}$$

Using the inequality

$$F_{n_{s+1}-m, \nu}(f, x)_{X^p} \leq \frac{1}{2} F_{n_s-m, \nu}(f, x)_{X^p} \quad \text{for } \begin{cases} \nu = 0, 1, 2, \dots, n_{s+1} - n_s - 1 \\ s = 0, 1, 2, \dots, t - 2 \end{cases}$$

we obtain

$$\begin{aligned}
& \sum_{s: n_{s+1} - n_s > \nu} [F_{n_s - m + \nu, m}(f, x)_{X^p} + F_{n_s - m + \nu, \nu}(f, x)_{X^p}] \\
= & F_{n_p - m + \nu, m}(f, x)_{X^p} + F_{n_p - m + \nu, \nu}(f, x)_{X^p} \\
& + \sum_{s \geq p+1: n_{s+1} - n_s > \nu} [F_{n_{s+1} - m + \nu, m}(f, x)_{X^p} + F_{n_{s+1} - m + \nu, \nu}(f, x)_{X^p}] \\
\leq & F_{n_p - m + \nu, m}(f, x)_{X^p} + F_{n_p - m + \nu, \nu}(f, x)_{X^p} \\
& + \sum_{s: s \geq p+1} F_{n_s - m, m}(f, x)_{X^p} [F_{n_s - m, m}(f, x)_{X^p} + F_{n_s - m, \nu}(f, x)_{X^p}] \\
\leq & F_{n_p - m + \nu, m}(f, x)_{X^p} + F_{n_p - m + \nu, \nu}(f, x)_{X^p} + 2 [F_{n_{p+1} - m, m}(f, x)_{X^p} + F_{n_{p+1} - m, \nu}(f, x)_{X^p}] \\
\leq & 3 [F_{n_p - m + \nu, m}(f, x)_{X^p} + F_{n_p - m + \nu, \nu}(f, x)_{X^p}] ,
\end{aligned}$$

where  $p$  denote the smallest index  $s$  having the property  $n_{s+1} - n_s > \nu$ .  
Hence

$$\begin{aligned}
\sum_{s=0}^{t-1} \left| \frac{1}{m+1} \tau_{n_s, m} f(x) \right| & \leq K \sum_{\nu=0}^{n+m-1} \frac{F_{n-m+\nu, m}(f, x)_{X^p} + F_{n-m+\nu, \nu}(f, x)_{X^p}}{m + \nu + 1} \\
& \leq K \sum_{\nu=0}^n \frac{F_{n-m+\nu, m}(f, x)_{X^p} + F_{n-m+\nu, \nu}(f, x)_{X^p}}{m + \nu + 1}.
\end{aligned}$$

and our proof follows.  $\blacksquare$

## 4.2 Proof of Theorem 3

The proof follows by the obvious inequality

$$\|E_n(f, x; \delta)_C\|_C \leq E_n(f)_C .$$

$\blacksquare$

## References

- [1] S. Aljančič, R. Bojanic and M. Tomić, On the degree of convergence of Fejér-Lebesgue sums, L'Enseignement Mathématique, Geneve, Tome XV (1969) 21-28.
- [2] P. L. Butzer, R.J. Nessel, Fourier analysis and approximation, Basel und Stuttgart 1971.
- [3] L. Leindler, Sharpening of Stečkin's theorem to strong approximation, Analysis Math. 16 (1990), 27-38.

- [4] W. Lenski, Pointwise best approximation and de la Vallée-Poussin means, submitted.
- [5] S. B. Stečkin, On the approximation of periodic functions by de la Vallée Poussin sums, *Analysis Math.* 4 (1978), 61-74.
- [6] N. Tanović-Miller, On some generalizations of the Fejér-Lebesgue theorem, *Boll. Un. Mat. Ital. B*(6) 1 (1982), no. 3, 1217-1233.